

Monday, November 24, Lecture 36: Euler-Lagrange

Midterm next Wednesday, Review Monday. Course Evaluations

Last time...

Particle moving in empty space:



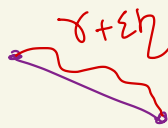
80%?,
get practice
final

Can find trajectory by minimizing arc length
 $I: \{\text{trajectories}\} \rightarrow \mathbb{R}$

$$I(\gamma) = \int_{t_1}^{t_2} F(t, \gamma, \dot{\gamma}) dt$$
$$= \int_{t_1}^{t_2} \|\dot{\gamma}\| dt$$

considered perturbations of paths: want stationary

$$0 = \left. \frac{dI}{d\varepsilon}(\gamma + \varepsilon\eta) \right|_{\varepsilon=0}$$



Lemma:

If $g: [0,1] \rightarrow \mathbb{R}^3$ has

$$0 = \int_0^1 \eta \cdot g dt$$

for every $\eta: [0,1] \rightarrow \mathbb{R}^3$, $\eta(0) = \eta(1) = 0$,

then g must be zero.

More generally:

Let

$$I(\epsilon) = \int_{t_1}^{t_2} F(t, \chi_\epsilon, \dot{\chi}_\epsilon) dt$$

think of $\chi_\epsilon = \chi + \epsilon \eta$



Then set

$$0 = \frac{dI}{d\epsilon} \Big|_{\epsilon=0} = \int_{t_1}^{t_2} \frac{\partial F}{\partial \chi_\epsilon} \cdot \frac{d\chi_\epsilon}{d\epsilon} + \frac{\partial F}{\partial \dot{\chi}_\epsilon} \frac{d\dot{\chi}_\epsilon}{d\epsilon} dt$$

$$\frac{d(\chi + \epsilon \eta)}{d\epsilon} = \eta \quad \rightarrow \quad \int_{t_1}^{t_2} \frac{\partial F}{\partial \chi_\epsilon} \cdot \eta + \frac{\partial F}{\partial \dot{\chi}_\epsilon} \cdot \dot{\eta} dt$$

$$\frac{d(\dot{\chi} + \epsilon \dot{\eta})}{d\epsilon} = \dot{\eta}$$

$$\begin{aligned} \text{IBP} \\ + \eta(t_1) = 0 \\ \eta(t_2) = 0 \end{aligned} \quad \rightarrow$$

$$= \int_{t_1}^{t_2} \frac{\partial F}{\partial \chi_\epsilon} \cdot \eta dt + \left(\frac{\partial F}{\partial \dot{\chi}_\epsilon} \cdot \eta \right) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{\chi}_\epsilon} \right) \cdot \eta dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial \chi_\epsilon} - \frac{d}{dt} \frac{\partial F}{\partial \dot{\chi}_\epsilon} \right) \cdot \eta dt$$

Lemma from last lecture $\Rightarrow$

Euler-Lagrange Eqn for $F(t, \chi, \dot{\chi})$,

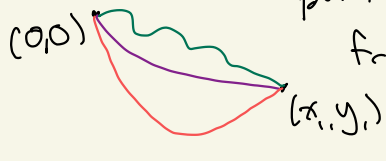
$$0 = \frac{\partial F}{\partial \chi} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{\chi}} \right)$$

Big idea: if we can find an F which encodes the physics of our system, Euler-Lagrange eqn allows us to find eqns of motion!

Ex: Brachistochrone Problem (aka the pinewood derby problem)

shortest time

Fix two points. Goal: find the curve γ on which a particle moving along γ gets from origin to p , the fastest.



- Fastest = minimize time,
- $\{\text{paths}\} = \{\text{graphs } y(x) \text{ with } y(0)=0 \text{ and } y(x)=y\}$

$$I = \int dt, \text{ and from rest} = \text{initial velocity} \rightarrow \text{zero}$$



arc length

$$v dt = ds$$

velocity

(no friction)

Let's use conservation of energy to help!

$$\text{kinetic: } \frac{1}{2} m v^2 = \frac{1}{2} m \left(\frac{ds}{dt} \right)^2$$

$$\text{potential: } -mgy$$

$$\text{Initial energy} = \text{kinetic} + \text{potential} = 0.$$

$$= \text{energy at any point } (x,y)$$

$$\frac{1}{2} m v^2 = mgy \Rightarrow v = \sqrt{2gy}$$

So, we want to minimize

$$I = \int dt = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2gy}} = \int_0^{x_1} \underbrace{\sqrt{\frac{1+y'^2}{2gy}}}_{F(x, y, y')} dx$$

$$ds = \sqrt{dx^2 + dy^2} \\ = \sqrt{1+y'^2} dx$$

Apply Euler-Lagrange:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$$

Notice: F is only indirectly a function of x :

$$F(x, y, y') = F(y, y'), \text{ but } y = y(x): \\ \text{and } y' = y'(x):$$

$$\Rightarrow 0 = \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = \frac{\partial F}{\partial y} - \left(0 + \frac{\partial F}{\partial y} \cdot y' + \frac{\partial F}{\partial y'} \cdot y'' \right)$$

multiply by y' ,

$$0 = \frac{\partial F}{\partial y} y' - \frac{\partial F}{\partial y} (y')^2 - \frac{\partial F}{\partial y'} y' y''$$

$$= \frac{d}{dx} \left(F - \frac{\partial F}{\partial y'} y' \right),$$

Since

$$\frac{d}{dx} \left(F - \frac{\partial F}{\partial y'} y' \right) = \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' - \left(\underbrace{\frac{\partial F}{\partial y}}_{\text{red}} \cdot y' + \underbrace{\frac{\partial F}{\partial y'} \cdot \frac{dy'}{dy} \cdot \frac{dy}{dx}}_{\text{red}} \right) y' \\ - \frac{\partial F}{\partial y'} y''$$

$\Rightarrow \quad \mathcal{O} = \frac{d}{dx} \left(F - \frac{dF}{dy'} y' \right) \leftarrow$ this trick is called the Beltrami identity. There are other tricks in Boas.

So

$$F - y' \frac{dF}{dy'} = C_1$$

$$\sqrt{\frac{1+y'^2}{2gy}} - \frac{y'}{\sqrt{2gy}} \cdot \frac{y'}{\sqrt{1+y'^2}} = C$$

$$\frac{1+y'^2}{2gy} - \frac{y'^2}{2gy} = C \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$$

$$1 = C \sqrt{(1+y'^2) 2gy}$$

$$\frac{1}{c^2 2g} = (1+y'^2) y$$

This isn't too bad of an ODE to solve.

Get the parametric eqns

$$x(t) = \frac{1}{2} \frac{1}{c^2 2g} (t - \sin t)$$

$$y(t) = \frac{1}{2} \frac{1}{c^2 2g} (1 - \cos t)$$

These describe a cycloid:

Animation:

https://en.wikipedia.org/wiki/Brachistochrone_curve

